Non-Classical Transport in Fractal Media as Applied to Radioactive Waste Problem: Anisotropic Random Advection Model - 11147

Leonid Bolshov[*,**], Peter Kondratenko[*,**], Leonid Matveev[*]
*Nuclear Safety Institute of Russian Academy of Sciences,
115191 Moscow, Russia
**Moscow Institute of Physics and Technology (State University),
141700 Moscow Region, Russia

ABSTRACT

Advection flow is a dominating physical mechanism of contaminant transport over geologic fractured media. Natural fractures often combine into percolation cluster, having fractal properties. As a result long-ranged correlations arise in the fluctuating velocity field of infiltration flux. The presence of preferred direction due to the gravity force leads to the strong anisotropy of the advection velocity field. Two basic dimensionless parameters $h > 0$ and $\beta > 1$ determine power exponents in the flux correlation spatial decay and anisotropy property, respectively. The values of these parameters are the key factors determining transport regimes. In the parameter domain expressed by inequalities $\beta(1+h) < 2$ and $1 < \beta < 2$, anisotropic super-diffusion regime is realized. When the inequalities $h < 1$, $\beta(1+h) > 2$ are valid, a compromise arises between two various transport regimes that is super-diffusion along the vertical direction and classical diffusion in horizontal plane. Concentration decay at long distances (in concentration tails) is of exponential type. Horizontal size of concentration signal undergoes to contraction at asymptotically large distances. Anomalous transport modes are realized at times when the contaminant plume sizes are much less than the correlation length, which is an upper bound of the fractality interval. At the later times contaminant transport occurs in a classical anisotropic diffusion-advection regime. The change of transport regimes in time results in a two-stage structure of concentration tails at later times. The nearest stage of the tail has classical Gaussian form and the remote one is of super-diffusive type.

INTRODUCTION

To perform reliable safety assessments of the radioactive waste disposals one needs to have models adequately describing radionuclide transport in geologic formations. When considering radionuclide migration in fractured rocks, advection due to moisture infiltration is the principal mechanism of transport over large distances. One of the key factors determining the transport characteristics in this case is in that infiltration paths formed by fracture networks represent percolation clusters which, as it is well known, have fractal properties. A problem of tracer transport in fractal media remains a subject of active interest for a long time [1-5]. The model of tracer advection in a random velocity field with long-range correlations reflects the main peculiarities of transport in such media.

This model was repeatedly studied in [6-9], where it has been shown that the tracer transport regime is superdiffusive for sufficiently slow power-law spatial decay of velocity correlation function. One more important conclusion of the theory [8, 9] is that concentration decay at large distances (in the “tail”) is of a contracted exponential type and is faster than a Gaussian one in

classical diffusion. Later on the influence of various factors such as finite value of correlation length [10] and the presence of traps [11] on the transport regimes in fractal media was analyzed.
A general assumption accepted in the studies [6-11] is the isotropy of fluctuating velocity field. However, the presence of preferred direction, determined e.g. by gravity vector, leads to anisotropic properties of the flow and what is more important to violation of the symmetry under the space inversion. Description of such flow in fractal media relates to the directed percolation problem, which was, for example, investigated in [12-14]. There it has been shown that power-law decay of correlation functions remains valid in the presence of anisotropy that eventually is the result of the self-similarity of fractal media. This allows us to use the ideas of critical phenomena theory, similarly to our previous work [8], but in more general formulation.
The aim of the present paper is to study the tracer transport over fractal medium in the anisotropic random advection model with long-range correlations. Special attention is paid to concentration behavior at large distances (in concentration tails).

PROBLEM FORMULATION

A basis of the random advection model is the equation for tracer particle concentration $c(\vec{r},t)$:

$$\frac{\partial c}{\partial t} + \vec{\nabla}(\vec{v}c) = 0 . \qquad \text{(Eq.1)}$$

Here t is a time, $\vec{\nabla}$ is a divergence operator and the advection velocity $\vec{v}(\vec{r})$ is a random function of coordinates, obeying the incompressibility condition

$$div\,\vec{v} = 0 , \qquad \text{(Eq.2)}$$

It is convenient to represent the advection velocity in the form

$$\vec{v}(\vec{r}) = \vec{u} + \vec{V}(\vec{r}), \qquad \text{(Eq.3)}$$

where $\vec{u} = \langle v(\vec{r}) \rangle$ is the mean value of the velocity and $\vec{V}(\vec{r})$ is its fluctuating part. $\langle ... \rangle$ means averaging over an ensemble of the medium realization.
The main feature of isotropic random fractal medium studied in [8] is the absence of any spatial scale in a large spatial interval limited above by a correlation length of the medium. In other words, the medium in this interval is self-similar. Taking advantage of the ideas of critical phenomenon theory [15] we considered transport processes to be scale invariant. This means that a macroscopic transport equation for averaged quantity A remain unchanged under simultaneous transformations

$$\vec{r} \to \lambda \vec{r} , \qquad \text{(Eq.4)}$$

and

$$A \to \lambda^{\Delta_A} A , \qquad \text{(Eq.5)}$$

where λ is a real positive parameter. The exponent Δ_A is termed as a scaling dimension of quantity A.
In the presence of preferred direction the scaling transformations (Eq.4), (Eq.5) should be generalized [5]. Namely, a separate scaling dimension should be prescribed to each coordinate. We will study our problem using Cartesian coordinates $\vec{r} = \{\vec{\rho}, z\}$ and $\vec{\rho} = (x, y)$, with Oz axis directed downward along the gravity vector.
Under the scaling transformation

$$\{\vec{\rho}, z\} \to \{\lambda^{1/\beta}\vec{\rho}, \lambda z\} \qquad \text{(Eq.6)}$$

a pair velocity correlation function $K_{zz}^{(2)}(\vec{r}_1,\vec{r}_2)=\langle V_z(\vec{r}_1)V_z(\vec{r}_2)\rangle$ for z - component of the fluctuating part of the velocity in the fractal interval, which will be defined later, obeys the relation

$$K_{zz}^{(2)}(\vec{r})\to\lambda^{-2h}K_{zz}^{(2)}(\vec{r}). \qquad \text{(Eq.7)}$$

Similar relationship with the replacement λ^{-2h} to λ^{-nh} is valid for z - components of n -point correlation functions.

Now h and β are exponents characterizing the random velocity field. According to Refs. [7,8], the tracer transport at $h>1$ is determined by short-range velocity distribution where instead of scale invariance a statistical homogeneity takes place and therefore the tracer transport corresponds to classical diffusion regime. In order to restrict ourselves with the non-trivial case in the present paper we consider $h<1$. For definiteness we also take $\beta>1$.

In accordance with definition, from (Eq.6) and (Eq.7) it follows that scaling dimensions of coordinates and zz -component of velocity correlation function are

$$\Delta_z=1,\quad \Delta_\rho=\frac{1}{\beta}, \qquad \text{(Eq.8)}$$

$$\Delta_{K_{zz}}=-2h. \qquad \text{(Eq.9)}$$

Taking into account scaling dimensions given by (Eq.8), (Eq.9) and also physical dimensions of the quantities entering the definition of the $K_{zz}^{(2)}(\vec{r})$ - function the later can be represented in the form

$$K_{zz}^{(2)}(\vec{r})\cong V^2\left(\frac{a_\parallel}{z}\right)^{2h}\varphi\left(\frac{\rho^\beta a_\parallel}{z\,a_\perp^\beta}\right), \qquad \text{(Eq.10)}$$

where $\rho=|\vec{\rho}|$, and $\varphi(x)$ is dimensionless function of self-similar variable. To ensure finiteness of $K_{zz}^{(2)}(\vec{r})$ - function along Oz axis ($\vec{\rho}\to 0$) and in $\vec{\rho}$ -plane ($z\to 0$) the function $\varphi(x)$ obeys the following asymptotic behavior:

$$\varphi(x\to 0)\to const,$$
$$\varphi(x\to\infty)\sim x^{-2h}. \qquad \text{(Eq.11)}$$

The other components of the pair correlation function can be constructed in a similar way using the scaling dimensions of velocity components, which are following from the incompressibility condition (Eq.2).

In an isotropic fractal medium the upper boundary of spatial interval of self-similar behavior of quantities like (Eq.10) is determined by a correlation length ξ of the medium. In an anisotropic medium two correlation lengths $\xi_\parallel$ and $\xi_\perp$ determine the upper boundaries in vertical and horizontal directions, respectively. Thus the expression (Eq.10) as well as similar ones for other components of the correlation function is valid in fractal interval:

$$a_\parallel << z << \xi_\parallel,\quad a_\perp << \rho << \xi_\perp, \qquad \text{(Eq.12)}$$

where $a_\parallel$ and $\xi_\parallel$, and $a_\perp$ and $\xi_\perp$ are low and upper boundaries of fractal interval in longitudinal and transverse directions, respectively.

From (Eq.12) in accordance with (Eq.8) the following relation is valid

$$\frac{\xi_{\parallel}}{a_{\parallel}} \approx \left(\frac{\xi_{\perp}}{a_{\perp}} \right)^{\beta} . \qquad \text{(Eq.13)}$$

At large distances, when $z >> \xi_{\parallel}$ or $\rho >> \xi_{\perp}$ the correlation functions decay exponentially.

For the mean velocity the following relation may be written [10]:

$$u \sim V \left(\frac{a_{\parallel}}{\xi_{\parallel}} \right)^{h} . \qquad \text{(Eq.14)}$$

In practice it is u that is determined by external conditions and, in turn, determines the fractality scale $\xi_{\parallel}$ of the medium.

Let us point out one important difference between our problem and the isotropic random advection model. In the isotropic model the components of correlation functions containing an odd number of z – components of the velocity under the space inversion transformation change their sign and thus should be zero due to the symmetry requirement. In our case the preferred direction determined by the gravity vector exists and the symmetry under the inversion of *Oz* direction violates. Thus the odd z-components of correlation function in general are nonzero.

MACROSCOPIC TRANSPORT EQUATIONS

Equation for tracer concentration averaged over ensemble of medium realization, $\overline{c}(\vec{r},t) = \langle c(\vec{r},t) \rangle$, has the standard form of particle number conservation law:

$$\frac{\partial \overline{c}}{\partial t} + div\, \vec{J} = 0 , \qquad \text{(Eq.15)}$$

where $\vec{J}(\vec{r},t)$ is the macroscopic particle number flux density. Taking the causality principle and linearity of the problem into account, we can write the expression for the flux density in the form

$$J_i(\vec{r},t) = -\int_{-\infty}^{t} dt' \int d\vec{r}' f_{ij}(\vec{r}-\vec{r}', t-t') \frac{\partial \overline{c}(\vec{r}',t')}{\partial r'_j} - \\ - \int_{-\infty}^{t} dt' \int d\vec{r}' f_i(\vec{r}-\vec{r}', t-t') \frac{\partial \overline{c}(\vec{r}',t')}{\partial t'} + u_i \overline{c}(\vec{r},t). \qquad \text{(Eq.16)}$$

Here response functions (integral kernels) obey the following symmetry properties: the absence of the invariance under the transformation $z \to -z$ and the isotropy in the $\vec{\rho}$ - plane. Thus, only three integral kernels in (Eq.13) are nonzero: f_{zz}, $f_{\rho\rho}$, f_z.

We consider the problem with the initial condition $\overline{c}(\vec{r},0) = c^{(0)}(\vec{r})$. Then the averaged tracer concentration can be expressed through its initial distribution by

$$\overline{c}(\vec{r},t) = \int G(\vec{r}-\vec{r}',t) c^{(0)}(\vec{r}') d\vec{r}' . \qquad \text{(Eq.17)}$$

Here $G(\vec{r},t)$ is the Green's function, the Fourier-Laplace transform of which according to (Eq.15) and (Eq.16) is

$$G_{\vec{k}p} = \left(p - M(\vec{k}, p) \right)^{-1} , \qquad \text{(Eq.18)}$$

where

$$M\left(\vec{k},p\right)=-k_i k_j\int_0^\infty dt\,e^{-pt}\int d\vec{r}\,e^{-i\vec{k}\vec{r}}\,f_{ij}\left(\vec{r},t\right)-ik_i p\int_0^\infty dt\,e^{-pt}\int d\vec{r}\,e^{-i\vec{k}\vec{r}}\,f_i\left(\vec{r},t\right)-ik_i u_i\,. \qquad \text{(Eq.19)}$$

SCALING ANALYSIS

The scaling dimension of the vertical component of the velocity follows from the definition of pair correlation function and (Eq.9):

$$\Delta_{v_z}=-h\,. \qquad \text{(Eq.20)}$$

Taking into account that under the transformation (Eq.6) a volume element changes as

$$d^3\vec{r}\to\lambda^{1+\frac{2}{\beta}}d^3\vec{r}\;, \qquad \text{(Eq.21)}$$

and the total number of particles in the medium

$$N_0=\int\bar{c}(\vec{r},t)d^3\vec{r}\;, \qquad \text{(Eq.22)}$$

remains constant, we arrive at the scaling dimensions of particle density and Green's function

$$\Delta_{\bar{c}}=\Delta_G=-\left(1+\frac{2}{\beta}\right). \qquad \text{(Eq.23)}$$

Considering the identity $\vec{J}=\left\langle\vec{v}c\right\rangle$, and using the (Eq.20), (Eq.23) the scaling dimensions of the flux density components are obtained as

$$\Delta_{J_z}=-\left(1+h+\frac{2}{\beta}\right),$$
$$\Delta_{J_\rho}=-\left(2+h+\frac{1}{\beta}\right). \qquad \text{(Eq.24)}$$

From here and from (Eq.15) the expression for scaling dimension of time follows

$$\Delta_t=1+h\,. \qquad \text{(Eq.25)}$$

And finally, using (Eq.16) for the integral kernels we have the expressions

$$\Delta_{f_z}=-\left(1+h+\frac{2}{\beta}\right),$$
$$\Delta_{f_{zz}}=-\left(1+2h+\frac{2}{\beta}\right), \qquad \text{(Eq.26)}$$
$$\Delta_{f_{\rho\rho}}=-\left(3+2h\right).$$

The scaling dimensions of Fourier-Laplace variables are

$$\Delta_p=-\Delta_t=-\left(1+h\right),\;\;\Delta_\kappa=-1\,,\;\;\Delta_q=-\frac{1}{\beta}\,, \qquad \text{(Eq.27)}$$

where we distinguish vertical κ and horizontal $\vec{q}$ components of the wave vector: $\vec{k}=\left(\vec{q},\kappa\right)$.

In accordance with the definition of the Fourier-Laplace transformation we obtain for scaling dimension of the Green's function and mass operator the following expressions

$$\Delta_{G_{pk}}=1+h\,,\;\;\Delta_{M_{pk}}=\Delta_p=-\left(1+h\right). \qquad \text{(Eq.28)}$$

Note that the results of this Section are valid only in the fractality interval defined by (Eq.12). In the opposite case tracer transport is described by the classical advection-diffusion equation.

TRACER CONCENTRATION BEHAVIOR

In this Section we are going to analyze concentration behavior at times when the tracer plume size does not exceed the value of correlation length ξ_α ($\alpha = \mathrm{II}, \perp$). As it follows from the results of the work [9], at these times tracer transport is determined by the fluctuating part of the velocity, and the drift due to the mean velocity $\vec{u}$ can be neglected. Thus in this section we omit the last term in the expression (Eq.19).

According to the results of the previous section the components of the integral kernel of the flux density in (Eq.16) take the form

$$f_z \sim \frac{V}{a_\parallel^3}\left(\frac{a_\parallel}{z}\right)^{1+h+2/\beta} \varphi_1(\eta, \varsigma), \qquad \text{(Eq.29)}$$

$$f_{zz} = \frac{V^2}{a_\parallel^3}\left(\frac{a_\parallel}{z}\right)^{1+2h+2/\beta} \varphi_2(\eta, \varsigma), \qquad \text{(Eq.30)}$$

$$f_{\rho\rho} = \frac{V^2}{a_\perp^3}\left(\frac{a_\parallel}{z}\right)^{3+2h} \varphi_3(\eta, \varsigma), \qquad \text{(Eq.31)}$$

$$\eta = \frac{z}{\left(a_\parallel^h Vt\right)^{\frac{1}{1+h}}}, \qquad \varsigma = \frac{\rho}{\left(a_\perp^{\beta(1+h)-1} Vt\right)^{\frac{1}{\beta(1+h)}}}, \quad \text{at} \quad z >> a_\parallel, \rho >> a_\perp \qquad \text{(Eq.32)}$$

and

$$f_z \sim \frac{V}{a_\parallel^3},\ f_{zz} \sim \frac{V^2}{a_{\mathrm{II}}^3},\ f_{\rho\rho} \sim \frac{V^2}{a_\perp^3} \quad \text{at} \quad z \le a_\parallel, \quad \rho \le a_\perp . \qquad \text{(Eq.33)}$$

In (Eq.29)-(Eq.31) the quantities $\varphi_\alpha(\eta, \varsigma)$ ($\alpha = 1,2,3$) are functions of two self-similar dimensionless variables.

Now we address the mass operator properties. According to (Eq.19) it can be written in the form

$$M\left(\vec{k}, p\right) = \kappa^2 M_\parallel + q^2 M_\perp + i\kappa p M_d , \qquad \text{(Eq.34)}$$

where $M_\parallel$, $M_\perp$, M_d are determined as Fourier-Laplace transform of the functions f_{zz}, $f_{\rho\rho}$, f_z, respectively.

It should be noted that in the case when the main area to contribute the integral of M_α corresponds to the distances $a << r << \xi$, M_α are scale invariant quantities with scale dimensions obeying the condition

$$\Delta_{M_\alpha} > 0 . \qquad \text{(Eq.35)}$$

The opposite inequality, $\Delta_{M_\alpha} < 0$, corresponds the condition that the integrals over spatial variables is determined by small distances $r \le a$. In this case we may put the exponential function in (Eq.19) equal to unity, so that the values of M_α will be constants and according to (Eq.34) the Green's function (Eq.18) takes the classical advection-diffusive form.

Further we will consider two cases: 1) $h < 1,\ 1 + h < 2/\beta$; 2) $h < 1,\ 1 + h > 2/\beta$.

1. $h < 1,\ 1 < \beta < \dfrac{2}{1+h}$

From (Eq.34) and (Eq.19) it follows

$$\Delta_{M_\parallel} = 1 - h \qquad \text{(Eq.36)}$$

$$\Delta_{M_\perp} = 2/\beta - 1 - h \qquad \text{(Eq.37)}$$

$$\Delta_{M_d} = 1. \qquad \text{(Eq.38)}$$

In this case the main contribution to integrals of (Eq.19) stems from fractality interval of spatial variables. The mass operator is scale-invariant quantity. Taking into account the values of scale dimensions (Eq.27) and (Eq.28) we arrive at the following representation for M

$$M = -p\psi\left(\vartheta, \vec{\chi}\right), \quad \vartheta = \kappa\left(\frac{a_\parallel^h V}{p}\right)^{\frac{1}{1+h}}, \quad \vec{\chi} = \vec{q}\left(\frac{a_\perp^{\beta(1+h)-1} V}{p}\right)^{\frac{1}{\beta(1+h)}}, \qquad \text{(Eq.39)}$$

where $\psi\left(\vartheta, \vec{\chi}\right)$ is a dimensionless function of two self-similar dimensionless variables. From the symmetry of our problem all quantities in Fourier-Laplace representation are isotropic in q-plane. Therefore we can consider M as a function of p, κ and $q = |\vec{q}|$:

$$M\left(\vec{k}, p\right) = \tilde{M}\left(\kappa, q; p\right). \qquad \text{(Eq.40)}$$

Using (Eq.39) and with taking into account (Eq.18) we can represent the Green's function

$$G\left(\vec{r}, t\right) = \int_{l-i\infty}^{l+i\infty} \frac{dp}{2\pi i} \int \frac{d\kappa d^2\vec{q}}{\left(2\pi\right)^3} \frac{e^{pt + i\kappa z + i\vec{q}\vec{\rho}}}{p - M\left(\vec{k}, p\right)} \qquad \text{(Eq.41)}$$

in the form

$$G\left(\vec{r}, t\right) = \left(a_\parallel^h V t\right)^{-\frac{1}{1+h}} \left(a_\perp^{\beta(1+h)-1} V t\right)^{-\frac{2}{\beta(1+h)}} g\left(\eta, \varsigma\right), \qquad \text{(Eq.42)}$$

where variables η and ς are defined by (Eq.32) and dimensionless function $g\left(\eta, \varsigma\right)$ possesses the property $g\left(\eta, \varsigma\right) \to 0$ if $\eta \to \infty$ or $\varsigma \to \infty$. From expression (Eq.42) we conclude that the tracer plume size in vertical and horizontal directions has the following time dependences

$$R_\parallel\left(t\right) \sim \left(a_\parallel^h V t\right)^{\frac{1}{1+h}}, \qquad \text{(Eq.43)}$$

$$R_\perp\left(t\right) \sim \left(a_\perp^{\beta(1+h)-1} V t\right)^{\frac{1}{\beta(1+h)}}. \qquad \text{(Eq.44)}$$

One can see that in the considered interval, $h < 1$ and $\beta < \dfrac{2}{1+h}$, the exponents $\dfrac{1}{1+h}$ and $\dfrac{1}{\beta\left(1+h\right)}$ are larger than $\dfrac{1}{2}$. Thus the tracer transport occurs in the super-diffusive regime [4,5] in all directions.

To derive a long-distance asymptotic of G function we need to know the analytical properties of the integrand in (Eq.41). As will be seen below, the main contribution into the integral of (Eq.41) at $\eta >> 1$ and/or $\varsigma >> 1$ is given by the area $\left|\operatorname{Im} p\right| << \operatorname{Re} p, \operatorname{Re} p > 0$. So, analyzing the analytical properties of the integrand in (Eq.41) we will accept $\operatorname{Im} p = 0,\ p > 0$. Consider the behavior of the mass operator in the two limiting cases: 1) $p \neq 0, \vec{k} \to 0$, 2) $p \to 0, \vec{k} \neq 0$.

1) $p \neq 0, \vec{k} \to 0$. In this case the convergence of the integrals on t in (Eq.19) is ensured by the multiplier e^{-pt}. The dependence of the mass operator on wave vectors is determined by a

decaying regime of the flux kernel f_{ik} at $r \to \infty$ at a fixed time. One can easily see that $f_{ik}(r,t)$ at large distances decreases faster than any negative-exponent power. Indeed, the very fact of the existence of arbitrary high-order velocity correlation function in our problem means that the velocity distributions determined by a functional that decreases at large velocities very steeply (in fact exponentially). In turn, this means that all power moments on coordinate of the $f_{ik}(r,t)$ function exist. Therefore the point $\vec{k}=0,\ p\neq 0$ is regular for the function $M\left(\vec{k},p\right)$ and we may write the following expression for the mass operator at $p\neq 0,\ \vartheta \ll 1,\ \chi \ll 1$:

$$M \sim -p\left(a_1 i\kappa \left(\frac{a_{\parallel}^{h} V}{p}\right)^{\frac{1}{1+h}} + a_2\kappa^2\left(\frac{V a_{\parallel}^{h}}{p}\right)^{\frac{2}{1+h}} + a_3 q^2\left(\frac{a_{\perp}^{\beta(1+h)-1}V}{p}\right)^{\frac{2}{\beta(1+h)}}\right), \qquad \text{(Eq.45)}$$

where the dimensionless constants a_α are of the order of unity.

2) $p\to 0,\ \vec{k}\neq 0$. Here two sub-cases are possible depending on the relation between κ and q.

At large longitudinal wave vector $\left(\frac{a_{\parallel} p}{V}\right)^{\frac{1}{1+h}} \ll \left(q a_{\perp}\right)^{\beta} \ll \kappa a_{\parallel}$ one has

$$\tilde{M}\left(\kappa,q;0\right) \sim -V a_{\parallel}^{h}\kappa^{1+h} f_1\left(\frac{\left(a_{\perp} q\right)^{\beta}}{a_{\parallel}\kappa}\right). \qquad \text{(Eq.46)}$$

In the opposite case $\left(\frac{a_{\parallel} p}{V}\right)^{\frac{1}{1+h}} \ll \kappa a_{\parallel} \ll \left(q a_{\perp}\right)^{\beta}$ the asymptotics of M is

$$\tilde{M}\left(\kappa,q;0\right) \sim -V a_{\perp}^{\beta(1+h)-1} q^{\beta(1+h)} f_2\left(\frac{a_{\parallel}\kappa}{\left(a_{\perp} q\right)^{\beta}}\right). \qquad \text{(Eq.47)}$$

The functions f_1 and f_2 approach constants when their arguments tend to zero.

From the asymptotic behavior of mass the operator at small values of wave vector an important difference from isotropic case follows. In isotropic case only even spatial moments of tracer concentration over each coordinate exist. For example, elementary calculations give the following expression for the second moment of concentration space distribution:

$$\left\langle z^2\right\rangle \equiv N_0^{-1}\int z^2 \bar{c}\left(\vec{r},t\right) dz\, d^2\rho = -\int \frac{\partial^2}{\partial\kappa^2} G_{\vec{k}p}\Big|_{\kappa=0,q=0} e^{pt}\frac{dp}{2\pi i} \sim \left(a^h V t\right)^{\frac{2}{1+h}} \sim R_{\parallel}\left(t\right)^2 \qquad \text{(Eq.48)}$$

Note that the first concentration moment in anisotropic model is nonzero. Evaluation similar to (Eq.48) with taking into account of (Eq.45) gives the estimate:

$$\left\langle z\right\rangle \sim R_{\parallel}\left(t\right) \sim \sqrt{\left\langle z^2\right\rangle}\,. \qquad \text{(Eq.49)}$$

Thus anomalous time-dependent drift of tracers takes place, so that the displacement of the tracer plume mass-center is described by the same time law as the square root of tracer dispersion. One can easily see that at times when the plume size is much smaller than correlation length, $t \ll t_\xi$, where

$$t_\xi \approx \frac{\xi_{\parallel}^{1+h}}{a^h V} = \frac{\xi_{\parallel}}{u}, \qquad \text{(Eq.50)}$$

the tracer displacement stipulated by this time-dependent drift is essentially larger than that determined by the drift with the mean velocity u

$$\langle z\rangle \sim \left(a^h Vt\right)^{\frac{1}{1+h}} >> ut\,, \qquad t << t_{\xi}\,. \qquad \text{(Eq.51)}$$

Let us now proceed to the Green's function (and, therefore, concentration) behavior at asymptotically large distances (in the tails). One can distinguish two regions (relatively large and small values of vertical coordinates) divided by a boundary $z=\tilde{Z}\left(\rho,t\right)$ in which the behavior of concentration tails is different. An expression for $z=\tilde{Z}\left(\rho,t\right)$ will be derived later.

Consider the domain $z>>\tilde{Z}\left(\rho,t\right)$. Supposing the value of q satisfying the inequality $\left(qa_{\perp}\right)^{\beta} << \kappa a_{\parallel}$ we will consider q as small real positive parameter. From the asymptotical expressions given by (Eq.45)-(Eq.47) we conclude (see, e.g., [8]) that $M\left(\vec{k},p\right)$ as a function of κ has a branch point lying for real p and q at the imaginary axis $i\kappa_b\left(p,q\right)$, in the vicinity of which the mass operator can be approximately represented as

$$\tilde{M}\left(\kappa,q;p\right) \sim -\frac{\kappa^2}{\left(\kappa - i\kappa_b\left(p,q\right)\right)^{\frac{1-h}{2}}}\,. \qquad \text{(Eq.52)}$$

Shifting the integration contour into the upper half-plane of the complex κ - variable we come to the conclusion that the integral over κ in (Eq.41) is determined by the residue in the nearest to the real axis pole $\kappa_0\left(p,q\right)$, which satisfies the equation

$$p-\tilde{M}\left(\kappa_0,q;p\right)=0\,. \qquad \text{(Eq.53)}$$

Taking into account the mass operator expansion over q at $\left(qa_{\perp}\right)^{\beta} << \left(\dfrac{a_{\parallel}p}{V}\right)^{\frac{1}{1+h}}$, we arrive at

$$\kappa_0\left(p,q\right) \approx i\left(Bp^{\frac{1}{1+h}} + Fp^{\frac{1-2/\beta}{1+h}}q^2\right), \qquad \text{(Eq.54)}$$

where B and F are real and positive constants of the order of unity determined by the solution of (Eq.53). Finally the integral in (Eq.41) within a pre-exponential factor takes the form

$$G \sim \int\frac{dp}{2\pi i}\iint\frac{d^2q}{\left(2\pi\right)^2}\exp\left(pt - Bp^{\frac{1}{1+h}}z - Fp^{\frac{1-2/\beta}{1+h}}z\,q^2 + i\vec{q}\vec{\rho}\right). \qquad \text{(Eq.55)}$$

The integral over q is Gaussian and thus is calculated exactly. The main contribution to its value is determined by q of the order of

$$q \sim \frac{\rho}{z}\,p_0^{\frac{2/\beta-1}{1+h}}\,. \qquad \text{(Eq.56)}$$

To perform integration over p the saddle-point technique was used and the important area of p was determined by

$$p_0 \sim t^{-1}\left(\frac{z}{R_{\parallel}\left(t\right)}\right)^{\frac{1+h}{h}}\,. \qquad \text{(Eq.57)}$$

Finally for concentration tail we come to an approximate expression

$$G \sim \exp\left(-\Gamma_{\parallel} - \left(\frac{\rho}{R_{\perp}(t)}\right)^{2} \Gamma_{\parallel}^{\frac{2}{\beta(1+h)}-1}\right), \qquad \text{(Eq.58)}$$

where $\Gamma_{\parallel} >> 1$ is determined by

$$\Gamma_{\parallel} \simeq \left(\frac{z}{R_{\parallel}(t)}\right)^{\frac{1+h}{h}}. \qquad \text{(Eq.59)}$$

Formula (Eq.58) is valid under the condition that the main values of κ and q determining the integrals satisfies inequality $\kappa a_{\parallel} >> (qa_{\perp})^{\beta}$. Substituting the estimates for κ from (Eq.54), q from (Eq.56) and p from (Eq.57) into the above inequality we conclude that the region of validity of (Eq.58) and (Eq.59) is

$$z >> \tilde{Z}(\rho,t) \approx \rho^{\beta}\left(\frac{R_{\perp}(t)}{\rho}\right)^{\frac{\beta(\beta-1)(1+h)}{(\beta-1)+\beta h}}. \qquad \text{(Eq.60)}$$

In the area $z << \tilde{Z}(\rho,t)$ we begin calculations in (Eq.41) with the integration over $\vec{q}$. Using relations between Bessel and Hankel functions [16] we get

$$G = \int_{l-i\infty}^{l+i\infty} \frac{dp}{2\pi i} \int_{-\infty}^{+\infty} \frac{d\kappa}{2\pi} \int_{-\infty+i0}^{+\infty+i0} \frac{qdq}{4\pi} \frac{e^{pt+i\kappa z} H_0^{(1)}(q\rho)}{p - M(\vec{k}, p)}. \qquad \text{(Eq.61)}$$

Then taking advantage of the asymptotics of $H_0^{(1)}(q\rho)$ for $q\rho >> 1$ and making successive integrations over q, κ and p we arrive (within a pre-exponential factor) at

$$G \sim \exp\left(-\Gamma_{\perp} - \left(\frac{z - R_{\parallel}(t)}{R_{\parallel}(t)}\right)^{2} \Gamma_{\perp}^{\frac{1-h}{1+h}}\right), \quad z << \tilde{Z}(\rho,t) \qquad \text{(Eq.62)}$$

where $\Gamma_{\perp} >> 1$ is determined as

$$\Gamma_{\perp} \sim \left(\frac{\rho}{R_{\perp}(t)}\right)^{\frac{\beta(1+h)}{\beta(1+h)-1}}. \qquad \text{(Eq.63)}$$

2. $\frac{2}{1+h} < \beta$

In this case from the condition $h < 1$ and the expression (Eq.36) it follows that the properties of $M_{\parallel}$ remains valid and its contribution to mass operator, as before, has scale-invariant form

$$M_{\parallel} \sim \kappa^{1+h} \tilde{f}\left(\frac{(Va_{\parallel}^{h}\kappa)^{1+h}}{p}, \frac{(a_{\perp}q)^{\beta}}{a_{\parallel}\kappa}\right). \qquad \text{(Eq.64)}$$

As to the integrals for $M_{\perp}$, the very fast decay of correlation functions with distance leads to that the main area to contribute the integrals corresponds to $r \le a$. Thus the expression for the mass operator takes the form

$$M \sim -A_{\parallel}\kappa^{1+h}\tilde{f}\left(\frac{\left(Va_{\parallel}^{h}\kappa\right)^{1+h}}{p},\frac{\left(a_{\perp}q\right)^{\beta}}{a_{\parallel}\kappa}\right)-A_{\perp}q^{2}. \qquad \text{(Eq.65)}$$

At first sight, the expression (Eq.65) contains a contradiction that scaling dimension of q determined by the second term in (Eq.65) differs from the scaling dimension following from (Eq.64). But one can easily see that under the condition $\frac{2}{\beta}<1+h$ in the wave vector area $\left(a_{\parallel}\kappa\right)^{1+h} >> \left(a_{\perp}q\right)^{2}$ (that is when the contribution of the first term in (Eq.65) dominates) the relation $\left(a_{\perp}q\right)^{\beta} << a_{\parallel}\kappa$ is valid. Hence it follows that corrections due to the second argument of $\tilde{f}$ function can be neglected.

From (Eq.65) it follows the tracer plume size temporal dependence in two directions

$$R_{\parallel}\left(t\right) \sim \left(a^{h}Vt\right)^{\frac{1}{1+h}}, \qquad \text{(Eq.66)}$$

$$R_{\perp} \sim \sqrt{A_{\perp}t}\,. \qquad \text{(Eq.67)}$$

The concentration behavior in tails in this case is also different in two spatial regions. In the domain $\left(a_{\parallel}\kappa\right)^{1+h} >> \left(a_{\perp}q\right)^{2}$ the “characteristic” equation (Eq.53) for the residue $\kappa_{0}\left(p,q\right)$ taking account of (Eq.65) has the form

$$1+\frac{A_{\perp}q^{2}}{p}+A_{\parallel}\frac{\kappa^{1+h}}{p}\tilde{f}\left(\frac{\left(Va_{\parallel}^{h}\kappa\right)^{1+h}}{p}\right)=0, \qquad \text{(Eq.68)}$$

whence, expanding the expression for the root of the (Eq.68) $\kappa_{0}\left(p,q\right)$ over small quantity $\frac{A_{\perp}q^{2}}{p}<<1$, we get

$$\kappa_{0}\left(p,q\right) \approx i\left(\tilde{B}p^{\frac{1}{1+h}}+\tilde{F}p^{-\frac{h}{1+h}}q^{2}\right). \qquad \text{(Eq.69)}$$

The integral over q is Gaussian and can be calculated exactly. Using the saddle-point technique for integration over p, we find that the long-distance asymptotic expression for the Green’s function up to a pre-exponential factor is

$$G \sim \exp\left(-\Gamma_{\parallel}-\left(\frac{\rho}{R_{\perp}\left(t\right)}\right)^{2}\right), \qquad \text{(Eq.70)}$$

where $\Gamma_{\parallel}$ is determined by the expression (Eq.59).

Taking into account that the dominant values of variables in integrals are of the order of

$$\kappa_{*} \sim p^{\frac{1}{1+h}},\ p_{*} \sim t^{-1}\left(\frac{z}{R_{\parallel}\left(t\right)}\right)^{\frac{1+h}{h}},\ q_{*} \sim \left(\frac{\rho}{R_{\perp}\left(t\right)}\right)^{2}, \qquad \text{(Eq.71)}$$

we conclude that (Eq.70) is valid in the domain determined by

$$\left(\frac{\rho}{R_{\perp}(t)}\right)^{2h} << \left(\frac{z}{R_{\parallel}(t)}\right)^{1+h}. \quad \text{(Eq.72)}$$

Similar calculations for the case $(a_{\parallel}\kappa)^{1+h} << (a_{\perp}q)^2$ lead to the result that in the region

$$\left(\frac{\rho}{R_{\perp}(t)}\right)^{2h} >> \left(\frac{z}{R_{\parallel}(t)}\right)^{1+h} \quad \text{(Eq.73)}$$

asymptotic behavior of concentration is described by

$$G \sim \exp\left(-\tilde{\Gamma}_{\perp} - \left(\frac{z}{R_{\parallel}(t)}\right)^{2} \tilde{\Gamma}_{\perp}^{\frac{1-h}{2(1+h)}}\right), \quad \text{(Eq.74)}$$

where

$$\tilde{\Gamma}_{\perp} \simeq \left(\frac{\rho}{R_{\perp}(t)}\right)^{2}. \quad \text{(Eq.75)}$$

TRACER TRANSPORT AT LARGE TIMES

As it follows from (Eq.43) and (Eq.44) at times $t >> t_{\xi}$, where t_{ξ} is determined by (Eq.50), the plume size exceeds the correlation length of the medium. The main area of integration in the expression (Eq.41) is concentrated near $p \sim t^{-1}$, $k \sim r^{-1}$, and thus in this case we are interested in the mass operator asymptotic for

$$\kappa << \xi_{\parallel}^{-1}, \quad q << \xi_{\perp}^{-1}, \quad \text{(Eq.76)}$$

Bearing in mind that at large scales $z >> \xi_{\parallel}$, $\rho >> \xi_{\perp}$ the velocity correlators as well as flux kernels (Eq.29), (Eq.30), (Eq.31) decay exponentially, in the domain of the wave vector values (Eq.76) we can estimate the quantities M_{α} introduced in (Eq.34) as follows:

$$M_{\parallel} = D_{\xi\parallel} \sim u\xi_{\parallel}. \quad \text{(Eq.77)}$$

$$M_{\perp} = D_{\xi\perp} \sim \xi_{\perp}^{2} t_{\xi}^{-1}. \quad \text{(Eq.78)}$$

$$M_{d} \sim p^{-1}V\left(\frac{a_{\parallel}}{\xi_{\parallel}}\right)^{h} \sim p^{-1}u. \quad \text{(Eq.79)}$$

where effective diffusivities $D_{\xi\parallel}$ and $D_{\xi\perp}$ are introduced and the relation for the mean advection velocity (Eq.14) is used.

In addition, at these times we have to take into account the last term in (Eq.19) which has the form of $i\kappa u$ and corresponds to the drift with constant velocity. One can easily see, that the contribution of $i\kappa p M_d$ from (Eq.79) is similar to that given by $i\kappa u$, and thus leads only to some renormalization of the mean drift velocity $u \to \tilde{u}$. As a result the mass operator takes the form

$$M \cong -i\kappa\tilde{u} - D_{\xi\parallel}\kappa^{2} - D_{\xi\perp}q^{2}, \quad \text{(Eq.80)}$$

and we come to the following expression for the Green's function

$$G \approx \frac{1}{(4\pi)^{3/2} D_{\xi\perp}\sqrt{D_{\xi\parallel}}\, t^{3/2}} \exp\left(-\frac{(z-\tilde{u}t)^{2}}{4D_{\xi\parallel}t} - \frac{\rho^{2}}{4D_{\xi\perp}t}\right). \quad \text{(Eq.81)}$$

In the tails the integral (Eq.41) is determined by the values of the wave vectors

$$\kappa \sim \frac{z}{2D_{\xi\parallel}t}, \quad q \sim \frac{\rho}{2D_{\xi\perp}t}, \qquad \text{(Eq.82)}$$

and at the distances

$$z >> \tilde{u}t, \quad \rho >> \frac{\xi_\perp}{t_\xi}t \qquad \text{(Eq.83)}$$

we return to the mass operator in the form (Eq.46) and (Eq.47). Thus at these distances concentration is determined by (Eq.58) and (Eq.62), resulting in two-stage structure of the concentration tail.

CONCLUSIONS

In summary, the influence of strong anisotropy on random advection in a fractal media was studied. We found that the transport regimes are determined by two parameters (h and β) characterizing advection velocity field.
When the decay of velocity correlation function is sufficiently slow, $h<1$, and flow anisotropy is moderate, $1<\beta<\frac{2}{1+h}$, super-diffusive transport takes place in all directions. Concentration decay at long distances (in concentration tails) is of exponential type. Transverse size of concentration signal undergoes to contraction at asymptotically large distances. One can say that tracer transport in concentration tails has the needle form.
For strong anisotropy, $\beta>\frac{2}{1+h}$, $h<1$, the transport regime in the vertical direction remains super-diffusive as before, but in horizontal plane it turns out to be of classical diffusion. The analysis of the tracer behavior in tails showed that it is non-classical in both directions.
At times when tracer plume size is less than correlation length an anomalous time-dependent drift occurs being much faster than one determined by advection with the mean velocity. The square of the displacement of plume mass-center is described by the same time-law as the dispersion of the tracer plume in the vertical direction.
At large times, when the tracer plume size exceeds the correlation length, the transport occurs in the classical regime of anisotropic advection-diffusion with the diffusivity tensor. The tails in this case consist of two stages with the nearest stage corresponding to classical regime and the remote one formed by earlier super-diffusive regime.

ACKNOWLEDGMENTS

We acknowledge support from the Russian Foundation for Basic Research (RFBR) under Projects 08-08-01009a, 09-08-00573a, and support from the Federal Target Program “Scientific and academic staff of innovative Russia” for the period 2009-2013 under Contract No. 02.740.11.0746, and support Savannah River Nuclear Solutions, LLC (USA) under Subcontract No. AC81967N.

REFERENCES

1. L. BOLSHOV, P. KONDRATENKO, K. PRUESS, and V. SEMENOV, “Nonclassical transport processes in geologic media: review of field and laboratory observations and basic physical concepts”, Vadose Zone Journal, V. 7, No 4, p. 1181 (2008).
2. E. BONNET, O. BOUR, N. E. ODLING, P. DAVY, I. MAIN, P. COWIE, B. BERKOWITZ, “Scaling of fracture systems in geological media”, Rev. Geophys., V. 39, p. 347 (2001).
3. B. B. MANDELBROT, “The Fractal Geometry of Nature”, San Francisco: W.H. Freeman (1982).
4. M.B. ISICHENKO, “Percolation, statistical topography, and transport in random media”, Reviews of Modern Physics, V. 64, No 4, p. 961 (1992).
5. D. BEN-AVRAHAM, S. HAVLIN, “Diffusion and Reactions in Fractals and Disordered Systems”, Cambridge University Press (2000).
6. D.L. KOCH, J.F. BRADY, “Anomalous diffusion in heterogeneous porous media”, Phys. Fluids, V. 31, p. 965 (1988).
7. D.L. KOCH, J.F. BRADY, “Anomalous diffusion due to long-range velocity fluctuations in the absence of a mean flow”, Phys. Fluids A, V. 1, p. 47 (1989).
8. A.M. DYKHNE, I.L. DRANIKOV, P.S. KONDRATENKO, L.V. MATVEEV, “Anomalous diffusion in a self-similar random advection field”, Physical Review E, V. 72, 061104 (2005).
9. I.L. DRANIKOV, P.S. KONDRATENKO, L.V. MATVEEV, “Anomalous transport regimes in a stochastic advection-diffusion model”, JETPh, V. 98, p. 945 (2004).
10. P.S. KONDRATENKO, L.V. MATVEEV, “Random advection in fractal medium with finite correlation length”, Physical Review E, V. 75, 051102 (2007).
11. L. BOLSHOV, P. KONDRATENKO, L. MATVEEV, K. PRUESS, “Elements of fractal generalization of dual-porosity model for solute transport in unsaturated fractured rocks”, Vadose Zone Journal, V. 7, No 4, p. 1198 (2008).
12. S. P. OBUKHOV, “The problem of directed percolation”, Physica A. V. 101, p.145 (1980).
13. J. L. CARDY, R.L. SUGAR, “Directed percolation and Reggeon field theory”, J. Phys.A: Math. Gen., V. 13, p. L423 (1980).
14. E.FREI, U. C. TÄUBER, F. SCHWABL, “Crossover from isotropic to directed percolation”, Physical Review E, V. 49, No 6, p. 5058 (1994).
15. S.-K. MA, “Morden theory of critical phenomena”, Benjamin, London (1976).
16. M. ABRAMOWITZ, I A. STEGUN, “Handbook of mathematical functions”, National Bureau of standards (1964).